SURFACES DERIVED FROM THE CUBIC VARIETY HAVING NINE DOUBLE POINTS IN FOUR DIMENSIONAL SPACE*

BY

VIRGIL SNYDER

Cubic varieties of ∞^3 points in space of four dimensions have been extensively studied by CASTELNUOVO † and by SEGRE, ‡ showing the close connection between the section of the enveloping cone by ordinary space (apparent contour) and the well known congruences of the second and third orders, with their focal surfaces. The treatment is exclusively synthetic and the methods there employed do not lend themselves readily to the discussion of particular cases. In the following paper I wish to call attention to two particular projections of the variety having nine double points. This variety is given a three page discussion in SEGRE's first paper, containing most of the results of Articles 1, 2, 3, 4, 6 of the present paper. The result of Art. 5 is stated by CASTELNUOVO in a footnote without proof, but no use is there made of it. The results in the remaining articles are new. I have made extensive use of all the papers cited, but as I shall proceed analytically, the development will be along different lines, and no knowledge of their contents will be assumed.

1. Let $\Gamma \equiv x_1 x_2 x_3 + \lambda x_4 x_5 x_6 = 0$ define a cubic variety in space of four dimensions S_4, where $\Sigma x_i = 0$. §

Every plane defined by $x_i = 0$ $(i = 1, 2, 3)$ and $x_k = 0$ $(k = 4, 5, 6)$ will lie entirely on Γ. These nine planes may be designated by α_{ik}. Any two of these having neither subscript in common will intersect in just one point which is a double point on Γ. Since two values of i and of k have been used to define each point, it may be denoted by A_{ik} with the remaining pair as subscripts. It follows immediately that Γ contains nine double points; through every double point pass four planes and in every plane lie four double points.

* Presented to the Society September 11, 1908.

† *Sopra una congruenza del* 3° *ordine e* 6^a *classe dello spazio a quattro dimensioni e sulle sue proiezioni nello spazio ordinario*, Atti del Istituto Veneto, series 6, vol. 5 (1887), pp. 1249–1281, and vol. 6 (1888), pp. 525–579.

‡ *Sulle varietà cubiche dello spazio a quattro dimensioni e su certi sistemi di rette e certe superficie dello spazio ordinario*, Memorie di Torino, series 2, vol. 39 (1889), pp. 1–48, and *Sulla varietà cubica con dieci punti doppi dello spazio a quattro dimensioni*, Atti di Torino, vol. 22 (1887), pp. 791–801.

§ From the results of Art. 21 of SEGRE's first paper it is easy to prove that the equation of any cubic variety with nine double points can be expressed in this form.

Any two planes α_{ik}, $\alpha_{ik'}$ having an index in common lie in the same S_3 defined by $x_i = 0$. Since the section of Γ made by $x_i = 0$ consists of three planes, the notation shows that this S_3 contains six of the nine double points, namely, all those not containing the subscript i. Through α_{ik} pass two fundamental spaces $x_1 = 0$ and $x_k = 0$. The first contains six double points and the second contains the four in α_{ik} and two others, not contained in the first. The double point not in either is A_{ik}, hence the S_3 determined by α_{ik} and A_{ik} contains no further double point.

2. If α, β, γ denote three independent parameters, it is seen that Γ may be generated by any one of the following trilinear systems:

$$\mathrm{I}\begin{cases}\alpha x_1 + \beta\lambda x_4 = 0,\\ \beta x_2 + \gamma x_5 = 0,\\ \gamma x_3 + \alpha x_6 = 0.\end{cases}\quad \mathrm{II}\begin{cases}\alpha x_1 + \beta\lambda x_5 = 0,\\ \beta x_2 + \gamma x_6 = 0,\\ \gamma x_3 + \alpha x_4 = 0.\end{cases}\quad \mathrm{III}\begin{cases}\alpha x_1 + \beta\lambda x_6 = 0,\\ \beta x_2 + \gamma x_4 = 0,\\ \gamma x_3 + \alpha x_5 = 0.\end{cases}$$

$$\mathrm{IV}\begin{cases}\alpha x_1 + \beta\lambda x_4 = 0,\\ \beta x_2 + \gamma x_6 = 0,\\ \gamma x_3 + \alpha x_5 = 0.\end{cases}\quad \mathrm{V}\begin{cases}\alpha x_1 + \beta\lambda x_5 = 0,\\ \beta x_2 + \gamma x_4 = 0,\\ \gamma x_3 + \alpha x_6 = 0.\end{cases}\quad \mathrm{VI}\begin{cases}\alpha x_1 + \beta\lambda x_6 = 0,\\ \beta x_2 + \gamma x_5 = 0,\\ \gamma x_3 + \alpha x_4 = 0.\end{cases}$$

Thus, Γ contains six different systems of ∞^2 straight lines. Any point of Γ will determine a line of each system uniquely, hence through every point of Γ pass six straight lines lying entirely on Γ. The section of Γ made by any S_3 will be a cubic surface F_3, and will therefore contain, in general, 27 lines. Of these, nine will be the intersections of S_3 and the planes α_{ik} and the remaining ones will be distributed equally among the six systems I, $\cdots$, VI. Thus each of these systems of lines is such that one line passes through every point of Γ and three lie in every S_3 cutting Γ.

The lines of I may be defined as the lines of Γ which cut α_{14}, α_{25}, α_{36}; similar triads are cut by each of the other systems.

3. The arrangement of the lines of Γ may be studied more closely by passing a pencil of spaces S_3 through one of the planes α_{ik}. For the sake of clearness, we shall choose α_{14} and follow the details for this case. The F_3 contained in an S_3 through α_{14} contains α_{14} as a factor, hence the remaining factor will define a quadric F_2. The section of F_2 in α_{14} is a conic through the four double points A_{25}, A_{26}, A_{35}, A_{36} through each of which passes a generator of each system. The points A_{25}, A_{26} lie in α_{34}; A_{35}, A_{36} in α_{24}; A_{25}, A_{36} in α_{16}; A_{26}, A_{36} in α_{15}; hence if the variable S_3 contains a point in any one of these planes, the quadric will break up into two planes, such as α_{34}, α_{24} or α_{15}, α_{16}. One other plane α_{ik} passes through each point: α_{35} through A_{26}, α_{36} through A_{25}, α_{26} through A_{35}, α_{25} through A_{36}. The pencil of quadrics will cut each of these planes in a pencil of lines, the vertices of the pencils of lines being at the double points.

The planes α_{25}, α_{36} intersect in A_{14} only, hence the lines (α_{25}, A_{36}), (α_{36}, A_{25}) belong to the same system of F_2; the lines (α_{26}, A_{35}), (α_{35}, A_{26}) belong to the other system. Hence the generators of F_2 belong to I and IV. A_{14} is the vertex of a cone, hence all its generators belong to I and IV. Thus, through each of the nine double points pass four plane pencils, one belonging to each of four systems I, $\cdots$, VI, and one quadric cone, belonging to each of the other two. Three quadric cones belong to each system.

4. The points A_{14}, A_{15}, A_{16} lie in the plane $x_2 = 0$, $x_3 = 0$;
" " A_{24}, A_{25}, A_{26} " " " " $x_1 = 0$, $x_3 = 0$;
" " A_{34}, A_{35}, A_{36} " " " " $x_1 = 0$, $x_2 = 0$;

and these three planes meet in the line $x_1 = 0$, $x_2 = 0$, $x_3 = 0$. Similarly, the three planes $A_{14}A_{24}A_{34}$, $A_{15}A_{25}A_{35}$, $A_{16}A_{26}A_{36}$ have the line $x_4=0$, $x_5=0$, $x_6=0$ in common.

The planes α_{14}, α_{15}, α_{16} meet in $(0, 1, -1, 0, 0, 0)$; α_{24}, α_{25}, α_{26} in $(1, 0, -1, 0, 0, 0)$; α_{34}, α_{35}, α_{36} in $(1, -1, 0, 0, 0, 0)$. These points all lie on the second of the preceding lines. Similarly, the points $\alpha_{14}\alpha_{24}\alpha_{34}$, $\alpha_{15}\alpha_{25}\alpha_{35}$, $\alpha_{16}\alpha_{26}\alpha_{36}$ lie on the first line $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.

The points A_{14}, A_{25}, A_{36} lie in the plane $x_1 + x_4 = 0$, $x_2 + x_5 = 0$;
" " A_{15}, A_{26}, A_{34} " " " " $x_1 + x_5 = 0$, $x_3 + x_4 = 0$;
" " A_{16}, A_{24}, A_{35} " " " " $x_3 + x_5 = 0$, $x_2 + x_4 = 0$;
" " A_{14}, A_{26}, A_{35} " " " " $x_1 + x_4 = 0$, $x_3 + x_5 = 0$;
" " A_{15}, A_{36}, A_{24} " " " " $x_1 + x_5 = 0$, $x_2 + x_4 = 0$;
" " A_{16}, A_{25}, A_{34} " " " " $x_2 + x_5 = 0$, $x_3 + x_4 = 0$.

All six of these planes pass through the point $P \equiv (1, 1, 1, -1, -1, -1)$.

5. The planes α_{14}, α_{25}, α_{36} do not lie in an S_3. They meet by twos in the points A_{14}, A_{25}, A_{36}. These three points determine the plane $x_1 + x_4 = 0$, $x_2 + x_5 = 0$. The pencil of spaces $l(x_1 + x_4) + m(x_2 + x_5) = 0$ having this plane for basis contains $\alpha_{14}(m = 0)$, $\alpha_{25}(l = 0)$, $\alpha_{36}(l = m)$. The lines of I are therefore transversals of this system. Now if the basis plane be connected with any point K in S_4, the resulting S_3 will also belong to this pencil. By varying λ we now obtain ∞^3 lines which cut the three planes α_{14}, α_{25}, α_{36}, and the fourth S_3 projectively. Now if the entire system be projected from K and cut by an S_3 not passing through K, the planes α_{14}, α_{25}, α_{36} project into planes not belonging to a pencil, and the fourth S_3 projects into the same plane as that into which $A_{14}A_{25}A_{36}$ projects. These planes form a tetrahedron, hence the ∞^3 lines of the family of system I for various values of λ all belong to a fixed tetrahedral complex. But by the same projection, the other systems will become tetrahedral complexes. These six tetrahedral complexes are independent of λ.

6. If from any point K in S_4 all the tangent lines be drawn to Γ and the projecting hypercone be cut by any S_3 not passing through its vertex, the resulting section, the apparent contour of Γ, will be a surface in S_3. The plane

through K and a line of Γ will cut Γ in a cubic curve consisting of the line and a conic. The conic cuts the line in two points, both of which are points of contact of the plane with Γ. Hence the lines of Γ project into bitangents of the apparent contour. Any line through K will cut Γ in three points, through each of which passes one line of each of the systems I, $\cdots$, VI. Any S_3 through K will cut Γ in an F_3 having three lines of each system, and this S_3 will cut the space of projection in three lines in a plane. Thus, the apparent contour is the focal surface of six different congruences, each of order and class three. These congruences will be represented by the usual symbol $(3, 3)$. The equation of the focal surface ϕ can be obtained by finding the locus of the point $(\xi) = (\xi_1, \cdots, \xi_6)$ on Γ such that the lines joining $K \equiv (a, b, c, d, e, f)$ to (x) and passing through (ξ) may intersect Γ in two coincident points. Let $\xi_1 = \mu a + \nu x_1$, etc. The binary cubic in μ, ν is of the form

$$\mu^3 \Gamma(a) + 3\mu^2\nu\Gamma(a, x) + 3\mu\nu^2\Gamma(x, a) + \nu^3\Gamma(x) = 0.$$

The discriminant is of the form $4H^3 + G^2 = 0$, in which H is the Hessian, and G the cubic covariant, of the binary form. The focal surface is therefore of order 6 and contains a cuspidal curve of order 6, the complete intersection of the quadric H and the cubic G. The nine double points A_{ik} will project into double points of ϕ. Each plane α_{ik} will become a double plane of ϕ, the conic of contact being the conic of intersection of the quadric of Γ in the space determined by K and α_{ik}. From Salmon's relations, ϕ must be of class 6. Hence: *The apparent contour ϕ of Γ is of order and class* 6, *has a cuspidal curve of order* 6, *nine double points and nine double planes. Four double points lie in each double plane, and four double planes pass through each double point. It is the focal surface of six congruences of order and class* 3, *each containing six quadric cones and six pencils of lines.*

Through each point of space pass three quadrics of the systems defined by the double planes, two of which must coincide for points on ϕ; hence ϕ is the envelope of nine different systems of quadrics, each of index three. Those belonging to α_{ik} pass through the double points in this plane and touch all the double planes passing through A_{ik}.

7. One variety of the pencil will pass through K, the center of projection. ϕ is now of order 4, and has six more double points lying in the tangent plane, the projection of the tangent S_3 to Γ at K. Every line through K will cut Γ in but two other points, hence the congruences in S_3 having ϕ for focal surface are of order 2 and class 3. One variety of the pencil will pass through P when $\lambda = 1$, and has a double point at that point. Γ now acquires five new double planes. The projection of the 10-nodal Γ from K will define the $(3, 2)$ congruence, dual of the preceding case. Both of these congruences are well known.*

* An excellent treatment of the 10-nodal Γ from the same point of view as that of SEGRE's paper is found in BERTINI: *Introduzione alla geometria proiettiva degli iperspazi*, 1907, pp. 176–187. Numerous references to the literature of related subjects are given.

8. In particular, let $K \equiv (a, b, c, 0, 0, 0)$, wherein $a + b + c = 0$, and let the S_3 of projection be the second polar of Γ as to K,

$$\frac{x_1}{a} + \frac{x_2}{b} + \frac{x_3}{c} = 0.$$

The equation of the focal surface now becomes

$$0 = 4a^2(c^2x_2^2 + bcx_2x_3 + b^2x_3^2)^3 - 27b^2c^2[a(cx_2 + bx_3)x_2x_3 + \lambda x_4x_5\{-a(cx_2 + bx_3) + bc(x_2 + x_3 + x_4 + x_5)\}]^2.$$

Thus the cuspidal sextic breaks up into two non-singular plane cubics intersecting in three points on the line $x_2 = 0$, $x_3 = 0$, which are points of inflexion on both. Moreover, the double planes coincide in sets of threes: the plane with which α_{14}, α_{15}, α_{16} coincide contains $A_{24}A_{25}A_{26}A_{34}A_{35}A_{36}$; the plane $\alpha_{24} \equiv \alpha_{25} \equiv \alpha_{26}$ contains $A_{14}A_{15}A_{16}A_{34}A_{35}A_{36}$; $\alpha_{34} \equiv \alpha_{35} \equiv \alpha_{36}$ contains $A_{14}A_{15}A_{16}A_{24}A_{25}A_{26}$. The conics of contact in the double planes are now replaced by three lines of contact.

The lines joining the nodal points by threes are:

$$x_5 = 0, \quad x_2\left(\frac{1}{a} - \frac{1}{b}\right) + x_3\left(\frac{1}{a} - \frac{1}{c}\right) + \frac{x_4}{a} = 0, \quad \text{containing } A_{14}, A_{24}, A_{34};$$

$$x_4 = 0, \quad x_2\left(\frac{1}{a} - \frac{1}{b}\right) + x_3\left(\frac{1}{a} - \frac{1}{c}\right) + \frac{x_5}{a} = 0, \quad \text{“} \quad A_{15}, A_{25}, A_{35};$$

$$x_4 = 0, \qquad x_5 = 0, \quad \text{“} \quad A_{16}, A_{26}, A_{36}.$$

They meet in the point

$$\left(\frac{1}{c} - \frac{1}{a}, \frac{1}{a} - \frac{1}{b}, 0, 0\right).$$

From the equation of ϕ we see that the section made by a plane containing any two of these lines is

$$4a^2(c^2x_2^2 + bcx_2x_3 + b^2x_3^2)^3 - 27a^2b^2c^2x_2^2x_3^2(cx_2 + bx_3)^2 = 0,$$

which is the same as

$$(cx_2 - bx_3)^2(2cx_2 + bx_3)^2(cx_2 + 2bx_3)^2 = 0.$$

It follows that each singular plane touches ϕ along three concurrent lines. The vertex of each triad is a point of inflexion on each of the cuspidal cubics lying on ϕ.

The distribution of points and lines is as follows:

In the plane $x_4 = 0$,

$$\frac{x_2}{b} - \frac{x_3}{c} = 0 \text{ joins } A_{15},\ A_{16};$$

$$\frac{x_2}{b} + \frac{2x_3}{c} = 0 \quad \text{“} \quad A_{25},\ A_{26};$$

$$\frac{2x_2}{b} + \frac{x_3}{c} = 0 \quad \text{“} \quad A_{35},\ A_{36}.$$

In the plane $x_5 = 0$,

$$\frac{x_2}{b} - \frac{x_3}{c} = 0 \text{ joins } A_{14},\ A_{16};$$

$$\frac{x_2}{b} + \frac{2x_3}{c} = 0 \quad \text{“} \quad A_{24},\ A_{26};$$

$$\frac{2x_2}{b} + \frac{x_3}{c} = 0 \quad \text{“} \quad A_{34},\ A_{36}.$$

In the plane $x_2\left(\frac{1}{a} - \frac{1}{b}\right) + x_3\left(\frac{1}{a} - \frac{1}{c}\right) + \frac{x_4}{a} + \frac{x_5}{a} = 0$,

$$\frac{x_2}{b} - \frac{x_3}{c} = 0 \text{ joins } A_{14},\ A_{15};$$

$$\frac{x_2}{b} + \frac{2x_3}{c} = 0 \quad \text{“} \quad A_{24},\ A_{25};$$

$$\frac{2x_2}{b} + \frac{x_3}{c} = 0 \quad \text{“} \quad A_{34},\ A_{35}.$$

The three triangles $A_{14}A_{15}A_{16}$, $A_{24}A_{25}A_{26}$, $A_{34}A_{35}A_{36}$ are therefore mutually in perspective, in such a way that lines joining corresponding points of any two pass through a point and the respective vertices of the remaining triangle, and the corresponding sides meet in three collinear points through which pass the respective sides of the remaining triangle.

If a, b or c vanishes, the corresponding points will lie on Γ for all values of λ. Thus all the varieties of the pencil will project from any one of these points into a plane counted six times. Similarly for points of intersection of $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ with Γ.

Any plane section of ϕ will be a sextic curve containing two collinear triads of cusps, hence every such section is a conical curve of genus 4.*

* These curves are characterized by a quadratic identity among three of the adjoint cubics. See the dissertation on the birational transformations of curves of genus 4 of Miss VAN BENSCHOTEN, Cornell, 1908.

9. It has been seen that the six systems I–VI belong to six fixed tetrahedral complexes. If now K be chosen in one of the basis planes, as $A_{14}A_{25}A_{36}$, this plane will project into a straight line, and all the lines cutting it will constitute a special linear complex, and the remaining lines of I will all belong to the remaining factor of the composite tetrahedral complex, which is in general a non-singular linear complex. Any two of the basis planes of the systems I–VI will meet in a point, but it has been seen that this point is

$$P \equiv (1,\ 1,\ 1,\ -1,\ -1,\ -1),$$

common to all six, hence if any two of the systems of bitangents to ϕ belong to linear complexes, then all six systems will belong to linear complexes.

Suppose now that the configuration be projected from P and cut by $x_5 = 0$. The systems I to VI belong to the following complexes, which are independent of λ:

I. $$p_{13} + p_{14} + p_{23} + p_{43} = 0;$$

II. $$p_{12} + p_{32} + p_{42} + p_{34} = 0;$$

III. $$p_{12} + p_{14} + p_{31} + p_{42} = 0;$$

IV. $$p_{12} + p_{14} + p_{32} + p_{42} = 0;$$

V. $$p_{13} + p_{32} + p_{42} + p_{34} = 0;$$

VI. $$p_{12} + p_{13} + p_{14} + p_{43} = 0.$$

Every complex of the net formed by the first three of these six is in involution with every one of the net formed by the second three.

The system of quadrics cut from Γ by the pencil of spaces defined by $x_1 + kx_4 = 0$ projects into the system

$$k^3(x_2+x_4)(x_3+x_4)+k^2[(x_2-x_1)(x_3+x_4)+(x_2+x_4)(x_3-x_1)+\lambda x_4(x_1+x_2+x_3+2x_4)]$$
$$+k[(x_2-x_1)(x_3-x_1)+\lambda(x_4-x_1)(x_2+x_3+x_4)-\lambda x_1(x_1+x_4)]-\lambda x_1(x_2+x_3+x_4)=0.$$

Through any point in space pass three quadrics of the system. The three generators of the first kind lie in a plane, the polar plane of the point in I, while the other three generators lie in the polar plane of IV. The envelope of this system of quadrics will be the focal surface and the basis plane counted twice. The equation of ϕ becomes

$$27(1-\lambda)\sigma_3^2 - 18(1-\lambda)\sigma_1\sigma_3(\sigma_2 - \lambda x_4\sigma_1 - \lambda x_4^2) + 4(1-\lambda)^2\sigma_1^3\sigma_3$$
$$-(1-\lambda)\sigma_1^2(\sigma_2 - \lambda x_4\sigma_1 - \lambda x_4^2)^2 + 4(\sigma_2 - \lambda x_4\sigma_1 - \lambda x_4^2)^3 = 0,$$

where σ_i is an elementary symmetric function of x_1, x_2, x_3 of weight i. The

coördinates of the nine double points are

$$A_{14} \equiv (1, 0, 0, -1);\ A_{24} \equiv (0, 1, 0, -1);\ A_{34} \equiv (0, 0, 1, -1);$$
$$A_{15} \equiv (0, 1, 1, -1);\ A_{25} \equiv (1, 0, 1, -1);\ A_{35} \equiv (1, 1, 0, -1);$$
$$A_{16} \equiv (1, 0, 0, 0);\quad A_{26} \equiv (0, 1, 0, 0);\quad A_{36} \equiv (0, 0, 1, 0);$$

and the equations of the nine double planes are

$$\alpha_{14} \equiv x_1 + x_4 = 0;\qquad \alpha_{24} \equiv x_2 + x_4 = 0;\qquad \alpha_{34} \equiv x_3 + x_4 = 0;$$
$$\alpha_{15} \equiv x_1 = 0;\qquad \alpha_{25} \equiv x_2 = 0;\qquad \alpha_{35} \equiv x_3 = 0;$$
$$\alpha_{16} \equiv x_2 + x_3 + x_4 = 0;\ \alpha_{26} \equiv x_3 + x_1 + x_4 = 0;\ \alpha_{36} \equiv x_1 + x_2 + x_4 = 0.$$

Thus, in addition to the four double points of the general case, each double plane α_{ik} contains the double point A_{ik}. The quadric cone having A_{ik} for vertex and belonging to two of the systems of bitangents now becomes the plane pencil $A_{ik}\alpha_{ik}$. Since this pencil belongs to two of the linear complexes, one of the directrices of the congruence defined by them lies in the plane α_{ik} and the other passes through the point A_{ik}.

The section made by any double plane consists of a conic counted twice and another conic. The former passes through four double points, and the latter through none. When $\lambda = -1$, each double conic breaks up into two lines intersecting in the point A_{ik}. Thus there are 18 lines which contain three double points. The point A_{ik} and the directrix of the linear congruence lying in α_{ik} are pole and polar with regard to the pencil of double conics.

The apparent contour of Γ *from* P *is a surface of order and class* 6, *has a cuspidal curve of order* 6, *nine double planes, and nine double points; through each double point pass five double planes, and in each double plane lie five double points. It has six systems of double tangents, each belonging to a linear complex. Through each point of space pass three bitangents belonging to each system. Each double point is the vertex of five pencils, one belonging to each of the four complexes, and the other to the remaining two. The surface is self dual in the six complexes.*

CORNELL UNIVERSITY,
July, 1908.